curve gives the solution for every case. Attention is also called to the fact that by this method an angle may be divided in any ratio or ratios in which a straight line may be divided. It is therefore as easily separated into parts having an incommensurable ratio as into any other, provided that that incommensurable ratio can be expressed by means of straight lines.

NOTE ON THE VOLUME OF A TETRAHEDRON IN TERMS OF THE COORDINATES OF THE VERTICES.

By DR. L. E. DICKSON.

1. Quite a variety of propositions of solid analytic geometry are needed for the usual derivation of the volume of a tetrahedron (cf. C. Smith, p. 24). If, as in the present note, we give an elementary proof making use merely of the concept of coordinates, we are in a position to apply the result to derive* easily several of the initial propositions in solid analytics, *e. g.*, that the equation of any plane is of the first degree, and conversely.

The plan of the proof (§ 3) is entirely obvious. The only novelty lies in a certain device which yields the result without computation. This device will first be illustrated in deriving the area of a triangle (§ 2).

2. Let the vertices of a triangle $\triangle$ taken in counter-clockwise order be (x_1, y_1), (x_2, y_2), (x_3, y_3). Then $\triangle$ can be expressed in terms of three right trapezoids with parallel sides y_i. The area of a right trapezoid with parallel sides y_1 and y_2, and base b, is $\frac{1}{2}b(y_1+y_2)$, being half of the rectangle of height y_1+y_2 and base b. Hence

$$2\triangle=(x_1-x_2)(y_1+y_2)+(x_2-x_3)(y_2+y_3)+(x_3-x_1)(y_3+y_1).$$

The device consists in setting $s=y_1+y_2+y_3$. Then

$$2\triangle=(x_1-x_2)(s-y_3)+(x_2-x_3)(s-y_1)+(x_3-x_1)(s-y_2).$$

Since each x occurs once positively and once negatively, the terms in s evidently cancel. The remaining terms give the expansion, according to the second column, of

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

*For plane analytics, this plan is followed in the chapter on graphic algebra in the writer's *College Algebra* (John Wiley and Sons).

3. Consider any tetrahedron $T=P_1P_2P_3P_4$, the notation for the vertices being chosen so that P_1 is above the plane of P_2, P_3, P_4, while the latter lie in counter-clockwise order when viewed from P_1. Denote P_i by (x_i, y_i, z_i); its projection on the x, y-plane is $Q_i=(x_i, y_i, 0)$. Now T can be expressed in terms of four truncated right triangular prisms $P_iP_jP_kQ_iQ_jQ_k$, i, j, k denoting three of the four numbers 1, 2, 3, 4. The area of $Q_iQ_jQ_k$ is given by a determinant (§ 2). Applying the formula (§ 4) for the volume of a truncated right prism, we get

$$6T=D_4(z_1+z_2+z_3)+D_3(z_1+z_2+z_4)+D_2(z_1+z_3+z_4)-D_1(z_2+z_3+z_4),$$

where

$$D_1=\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix},\ D_2=\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix},\ D_3=\begin{vmatrix} x_1 & y_1 & 1 \\ x_4 & y_4 & 1 \\ x_2 & y_2 & 1 \end{vmatrix},\ D_4=\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

The device consists in setting $s=z_1+z_2+z_3+z_4$. Then

$$6T=D_4(s-z_4)+D_3(s-z_3)+D_2(s-z_2)-D_1(s-z_1).$$

Here the terms free of s equal the expansion, according to the third column, of

$$D\equiv\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

The terms multiplying s are derived from the others by replacing each z by -1 and hence equal the expansion of a determinant derived from $-D$ by replacing each z by 1. But a determinant with two columns alike vanishes identically. Hence $T=\frac{1}{6}D$.

4. The volume of a truncated right prism P, whose base is a triangle $\triangle$, and lateral edges are a, b, c, is $\frac{1}{3}(a+b+c)\triangle$. This may be proved as in the geometries, or very simply as follows. Let $a \geqq b \geqq c$. Let the edge c be DE, d the side of triangle opposite D, h the perpendicular from D to d, so that $\triangle=\frac{1}{2}hd$. The plane through E parallel to $\triangle$ divides P into a right prism of volume $c\triangle$ and a pyramid with summit E, altitude h, and base a trapezoid with parallel sides $a-c$, $b-c$, and common perpendicular d. The area of the trapezoid is $\frac{1}{2}d(a+b-2c)$; the volume of the pyramid is therefore $\frac{1}{3}\triangle(a+b-2c)$. Adding $c\triangle$ to the latter, we get $P=\frac{1}{3}\triangle(a+b+c)$.

To give another proof, extend b (upwards) the length $a-b$. Thus to P we add a triangular pyramid with summit E, and base a right triangle of lege $a-b$ and d, and hence of volume $\frac{1}{2}d(a-b).\frac{1}{3}h=\frac{1}{3}\triangle(a-b)$. Next, extend c the length $a-c$. We thus add on a pyramid of summit E, and base equal to $\triangle$, and hence of volume $\frac{1}{3}\triangle(a-c)$. By these additions, P becomes a right prism of volume $\triangle a$. Hence

$$P+\tfrac{1}{3}\triangle[a-b]+\tfrac{1}{3}\triangle[a-c]=\triangle a,\quad P=\tfrac{1}{3}\triangle[a+b+c].$$